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The Effect of Changes in the Stability Derivatives on the Dynamic Behaviour of a Torpedo

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Summary.—This report investigates the extent to which the dynamic behaviour of a torpedo is sensitive to changes in its stability derivatives. The main object in carrying out the investigation was to provide guidance on the accuracy of measurement of the stability derivatives that should be necessary for any given torpedo. The considerations of the report are, however, also pertinent to the problem of deciding the effectiveness of possible changes in the design of a torpedo, the dynamic behaviour of which is unsatisfactory. Illustrative examples are worked out in detail. The report emphasises the importance of the so-called margin of stability.

1. *Introduction.*—The purpose of this report is to investigate the extent to which the dynamic behaviour of a torpedo is sensitive to changes in its stability derivatives. Since dynamic behaviour covers the whole class of possible motions of a torpedo, attention has had to be confined to certain well defined aspects. The main object in carrying out the investigation was to provide guidance on the accuracy of measurement of the stability derivatives that would be necessary for any given torpedo: specifically, what error in predicted performance will given errors in the stability derivatives cause? The considerations of the report are, however, also pertinent to the problem of deciding the effectiveness of possible changes in the design of a torpedo, the dynamic behaviour of which is unsatisfactory.

2. *The Motion of the Torpedo.*—We consider motion in a vertical plane only, and neglect buoyancy and trim effects. The treatment applies equally to motion in a horizontal plane only.

The relevant equations of motion are

$$Z_\alpha \alpha + Z_q \dot{\theta} + Z_{\delta_e} \delta_e = m_2 V \dot{\alpha} - m_1 V \dot{\theta} , \quad \ldots \quad (1)$$

$$M_\alpha \alpha + M_q \dot{\theta} + M_{\delta_e} \delta_e = J \ddot{\theta} , \quad \ldots \quad (2)$$

where

V	speed of torpedo, assumed constant
α	angle of attack
θ	pitch angle
δ_e	elevator angle
q	$\dot{\theta}$ pitching rate
Z	denotes the coefficient of a force normal to the torpedo axis

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M		denotes the coefficient of a moment about the transverse horizontal axis through the torpedo c.g.
Z_α	$=$	$\partial Z/\partial\alpha$, etc.
m_2		total transverse mass of torpedo $\simeq m + K_2m_f$
m_1	$=$	total longitudinal mass of torpedo $\simeq m + K_1m_f$
m	$=$	mass of torpedo
m_f	$=$	mass of displaced fluid
J_y	$=$	total moment of inertia about the transverse horizontal axis through the c.g. $\simeq I_y + K'I_{yf}$
I_y	$=$	moment of inertia of torpedo about the transverse horizontal axis through the c.g.
I_{yf}	$=$	moment of inertia of displaced fluid about the transverse horizontal axis through the c.g.
K', K_1, K_2		are Lamb's inertia coefficients for an equivalent ellipsoid.

The positive senses of the various parameters are illustrated in Fig. 1.

If we multiply each term of equations (1) and (2) by e^{-pt} and integrate with respect to the time t between 0 and ∞ throughout (denoting Laplace-transformed quantities by a bar) and eliminate $\bar\theta$ we have

$$[m_2VJ_yp^2 - (J_yZ_\alpha + m_2VM_q)p + M_qZ_\alpha - M_\alpha(m_1V + Z_q)]\bar\alpha$$
$$= [J_yZ_{\delta_e}p + M_{\delta_e}(m_1V + Z_q) - M_qZ_{\delta_e}]\bar\delta_e \quad \ldots \quad (3)$$

If we had found, instead, the equation connecting $\bar\xi_0$ or $p\bar\theta$ with $\bar\delta_e$, the left-hand side would have been identical with that of equation (3). We write this left-hand side as

$$\left.\begin{aligned} &[A_1p^2 + A_2p + A_3]\bar\alpha\,, \\ \text{where}\quad &A_1 = m_2VJ_y \\ &A_2 = -J_yZ_\alpha - m_2VM_q \\ &A_3 = M_qZ_\alpha - M_\alpha(m_1V + Z_q) \end{aligned}\right\} \quad \ldots \quad (4)$$

It follows from equation (3) that the transient part of the solution for $\alpha(t)$ will be

$$\lambda_1 e^{\mu_1 t} + \lambda_2 e^{\mu_2 t}\,, \quad \ldots \quad (5)$$

where μ_1 and μ_2, the decay constants of the motion are the roots of

$$A_1\mu^2 + A_2\mu + A_3 = 0 \quad \ldots \quad (6)$$

and λ_1 and λ_2 are constants.

In particular, if the elevators are locked at zero, the right-hand side of equation (3) disappears, and the expression (5) represents the complete solution for the angle of attack α, following a disturbance.

A torpedo is said to have dynamic stability, if, when disturbed from a straight-line path, it will again settle down to a straight-line path (but not necessarily the original straight-line path), that is, it tends to reduce its angle of attack to zero. If a dynamically unstable torpedo is disturbed from its straight-line path, it will circle with smaller and smaller radius until the linear analysis used here no longer applies. It is clear from equation (5) that the necessary and sufficient condition for the torpedo to have dynamic stability is that the roots of equation (6) have negative real parts. The necessary and sufficient condition for this is that A_1, A_2 and A_3 all have the same sign :

$$A_1 = m_2 V J_y > 0$$

$$A_2 = -J_y Z_\alpha - m_2 V M_q > 0$$

since $Z_\alpha < 0$, $M_q < 0$ for all conventional torpedoes. The criterion for dynamic stability is therefore that $A_3 > 0$. Since $Z_\alpha M_q > 0$, we can write

$$G = 1 - \frac{M_\alpha(m_1 V + Z_q)}{Z_\alpha M_q} > 0 \text{ for dynamic stability} . \quad .. \quad .. \quad (7)$$

G is called the margin of stability. The following Table indicates torpedo behaviour for different values of G.

Stability	G	Controllability	Application
Dynamically unstable .. Marginally stable ..	<0 0	Requires special control equipment ..	No known application.
Dynamically stable ..	0·1 0·2 0·3 0·4	Turns rapidly with small rudders ; hard to control and maintain in straight flight.	Homing torpedoes.
	0·5 0·6	Turns rapidly with medium-sized rudders ; controls moderately well.	Homing torpedoes and straight-running torpedoes.
	0·7 0·8	Turns rapidly with large rudders ; controls easily.	Straight-running torpedoes.
	0·9 1·0 $>$1·0	Requires very large rudders ; controls very easily.	Straight-running torpedoes.

2.1. *Circling Motion.*—Suppose the torpedo is moving steadily in a vertical circle of constant radius R, with the following (constant) values of its parameters

$$q = \dot\theta = \dot\theta^* ; \quad \alpha = \alpha^* ; \quad \delta_e = \delta_e^*$$

$$\dot\alpha = \ddot\theta = 0 .$$

Putting these values in equations (1) and (2) and solving for $\dot\theta^*$ and α^* we have

$$G \frac{\dot\theta^*}{\delta_e^*} = \frac{M_\alpha Z_{\delta_e} - Z_\alpha M_{\delta_e}}{Z_\alpha M_q} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (8)$$

$$G \frac{\alpha^*}{\delta_e^*} = \frac{M_{\delta_e}(m_1 V + Z_q) - M_q Z_{\delta_e}}{Z_\alpha M_q} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (9)$$

(We note that, since the right-hand side of equations (8) and (9) are both negative for all conventional torpedoes,

$$\operatorname{sgn} \frac{\dot\theta^*}{\delta_e^*} = \operatorname{sgn} \frac{\alpha^*}{\delta_e^*} = - \operatorname{sgn} G .$$

This implies that a dynamically stable torpedo ($G > 0$) turns with its elevators, while a dynamically unstable one ($G < 0$) turns against its elevators.)

In a stable ($G > 0$) turn of constant radius R, $V = R\dot{\theta}^*$, and from equation (8) we have

$$R = \frac{VGZ_\alpha M_q}{M_\alpha Z_{\delta_e} - Z_\alpha M_{\delta_e}} \frac{1}{\delta_e^*} \quad \ldots \quad (10)$$

3. *The Effect of Errors in the Stability Derivatives.*—We can now study the effects of errors in the stability derivatives Z_α, M_α, Z_{δ_e}, M_{δ_e}, Z_q and M_q on three aspects of the dynamic behaviour of a torpedo :

(*a*) The effect on the radius of turn R for a given elevator angle δ_e^*

(*b*) The effect on the margin of stability G

(*c*) The effect on the transient motion of the torpedo following a disturbance. This is done by studying the effect on the decay constants μ_1 and μ_2 defined by equation (5).

Errors in the range $\pm$ 20 per cent will be considered for the static and control surface derivatives Z_α, M_α, Z_{δ_e} and M_{δ_e}, and errors in the range $\pm$ 50 per cent for the rotary derivatives Z_q and M_q. Each case will be illustrated by examples of two torpedoes of widely differing hydrodynamic characteristics, Torpedo A (G about 1·0), and Torpedo B (G about 0·6). They have the following hydrodynamic coefficients :

TORPEDO A.

$$\frac{\partial C_L}{\partial \alpha} = -3 \cdot 09 ; \qquad \frac{\partial C_L}{\partial \delta_e} = -0 \cdot 70 ; \qquad \frac{\partial C_L}{\partial (l/R)} = -1 \cdot 40_5$$

$$\frac{\partial C_M}{\partial \alpha} = -0 \cdot 05_6 ; \qquad \frac{\partial C_M}{\partial \delta_e} = 0 \cdot 37 ; \qquad \frac{\partial C_M}{\partial (l/R)} = -0 \cdot 63_6 .$$

We use the relations

$$Z_\alpha = \tfrac{1}{2}\rho A V^2 \frac{\partial C_L}{\partial \alpha} ; \qquad Z_{\delta_e} = \tfrac{1}{2}\rho A V^2 \frac{\partial C_L}{\partial \delta_e} ; \qquad Z_q = \tfrac{1}{2}\rho A V l \frac{\partial C_L}{\partial (l/R)}$$

$$M_\alpha = \tfrac{1}{2}\rho A V^2 l \frac{\partial C_M}{\partial \alpha} ; \qquad M_{\delta_e} = \tfrac{1}{2}\rho A V^2 l \frac{\partial C_M}{\partial \delta_e} ; \qquad M_q = \tfrac{1}{2}\rho A V l^2 \frac{\partial C_M}{\partial (l/R)}$$

where ρ = density of water = 2 slugs/cu ft

A = maximum cross-sectional area of torpedo = 2·4 ft^2

V = speed of torpedo = 40 ft/sec

l = length of torpedo = 14 ft.

This gives

$$\frac{Z_\alpha}{10^3} = -11 \cdot 866 ; \qquad \frac{Z_{\delta_e}}{10^3} = -2 \cdot 688 ; \qquad \frac{Z_q}{10^3} = -1 \cdot 888$$

$$\frac{M_\alpha}{10^3} = -2 \cdot 957 ; \qquad \frac{M_{\delta_e}}{10^3} = -19 \cdot 891 ; \qquad \frac{M_q}{10^3} = -11 \cdot 967 .$$

Also m = mass of torpedo = 58·5 slugs

I_y = moment of inertia of torpedo about the transverse horizontal axis through the c.g. = 745 slugs/ft^2.

The Lamb inertia coefficients for an ellipsoid of the same fineness ratio (8) are

$$K_1 = 0\cdot029\,; \qquad K_2 = 0\cdot945\,; \qquad K' = 0\cdot840, \text{ giving}$$

$$\frac{m_1 V}{10^3} = +2\cdot408\,; \qquad \frac{m_2 V}{10^3} = +4\cdot551\,; \qquad \frac{J_v}{10^3} = +1\cdot371\,.$$

<u>TORPEDO B.</u>

$$\frac{\partial C_L}{\partial \alpha} = 2\cdot29\,; \qquad \frac{\partial C_L}{\partial \delta_e} = -0\cdot396\,; \qquad \frac{\partial C_L}{\partial (l/R)} = -1\cdot04$$

$$\frac{\partial C_M}{\partial \alpha} = +0\cdot556\,; \qquad \frac{\partial C_M}{\partial \delta_e} = 0\cdot229\,; \qquad \frac{\partial C_M}{\partial (l/R)} = -0\cdot50$$

$$\rho = 2 \text{ slugs/cu ft} \qquad m = 94\cdot47 \text{ slugs}$$

$$A = 2\cdot405 \text{ ft}^2 \qquad I_y = 1886\cdot8 \text{ slugs/ft}^2$$

$$V = 49 \text{ ft/sec} \qquad \text{Fineness ratio} = 11\cdot7, \text{ whence}$$

$$l = 20\cdot49 \text{ ft} \qquad K_1 = 0\cdot019\,; \qquad K_2 = 0\cdot968\,; \qquad K' = 0\cdot908\,.$$

These give

$$\frac{Z_w}{10^3} = 13\cdot223\,; \qquad \frac{Z_{\delta_e}}{10^3} = 2\cdot287\,; \qquad \frac{Z_q}{10^3} = 2\cdot511$$

$$\frac{M_w}{10^3} = +65\cdot785\,; \qquad \frac{M_{\delta_e}}{10^3} = 27\cdot095\,; \qquad \frac{M_q}{10^3} = 24\cdot738$$

$$\frac{m_2 V}{10^3} = +9\cdot110\,; \qquad \frac{m_1 V}{10^3} = +4\cdot717\,; \qquad \frac{J_v}{10^3} = +3\cdot600\,.$$

3.1. *The Effect on Radius of Turn.*—For a given elevator deflection δ_e^*, the radius of turn is (equation (10)).

$$R = R' \frac{V}{\delta_e^*}$$

where

$$R' = \frac{G Z_w M_q}{M_w Z_{\delta_e} - Z_w M_{\delta_e}} = \frac{Z_w M_q - M_w(m_1 V + Z_q)}{M_w Z_{\delta_e} - Z_w M_{\delta_e}} \qquad \ldots \quad \ldots \quad \ldots \quad (11)$$

We denote by R_0 and R_0' the values of R and R' when there are no errors in the stability derivatives, and by δR and $\delta R'$ the changes in R and R' due to changes δC in C, where C is one of Z_w, M_w, Z_{δ_e}, M_{δ_e}, Z_q and M_q.

Since V and δ_e^* are constant, it is clear that

$$\frac{\delta R}{R} = \frac{\delta R'}{R'}\,.$$

The fractional change in R for any given fractional error in C can be calculated from equation (11) as set down below, for all six interpretations of C.

We note that R_0 has the following values for the two torpedoes chosen as examples :

Torpedo A $R_0 = 144$ ft when $\delta_e^* = 10$ deg

Torpedo B $R_0 = 100$ ft when $\delta_e^* = 10$ deg.

Errors in Z_α

$$Z_\alpha \to Z_\alpha + \delta Z_\alpha$$

$$\frac{\delta R}{R} = \frac{M_\alpha[M_q Z_{\delta_e} - M_{\delta_e}(m_1 V + Z_q)]}{M_q Z_\alpha - M_\alpha(m_1 V + Z_q)} \; \frac{\delta Z_\alpha / Z_\alpha}{M_\alpha \dfrac{Z_{\delta_e}}{Z_\alpha} - M_{\delta_e}\left(1 + \dfrac{\delta Z_\alpha}{Z_\alpha}\right)}$$

Torpedo A : $$\frac{\delta R}{R} = \frac{\delta Z_\alpha / Z_\alpha}{-21{\cdot}95 - 22{\cdot}71 \dfrac{\delta Z_\alpha}{Z_\alpha}}$$

Torpedo B : $$\frac{\delta R}{R} = \frac{\delta Z_\alpha / Z_\alpha}{+0{\cdot}92 - 0{\cdot}64 \dfrac{\delta Z_\alpha}{Z_\alpha}}.$$

Errors in M_α

$$M_\alpha \to M_\alpha + \delta M_\alpha$$

$$\frac{\delta R}{R} = \frac{Z_\alpha[M_q Z_{\delta_e} - M_{\delta_e}(m_1 V + Z_q)]}{M_q Z_\alpha - M_\alpha(m_1 V + Z_q)} \; \frac{\delta M_\alpha / M_\alpha}{Z_\alpha \dfrac{M_{\delta_e}}{M_\alpha} - Z_{\delta_e}\left(1 + \dfrac{\delta M_\alpha}{M_\alpha}\right)}$$

Torpedo A : $$\frac{\delta R}{R} = \frac{\delta M_\alpha / M_\alpha}{+21{\cdot}95 - 0{\cdot}77 \dfrac{\delta M_\alpha}{M_\alpha}}$$

Torpedo B : $$\frac{\delta R}{R} = \frac{\delta M_\alpha / M_\alpha}{-0{\cdot}92 - 0{\cdot}27 \dfrac{\delta M_\alpha}{M_\alpha}}.$$

Errors in Z_{δ_e}

$$Z_{\delta_e} \to Z_{\delta_e} + \delta Z_{\delta_e}$$

$$\frac{\delta R}{R} = \frac{\delta Z_{\delta_e} / Z_{\delta_e}}{\dfrac{M_{\delta_e}}{M_\alpha} \dfrac{Z_\alpha}{Z_{\delta_e}} - 1 - \dfrac{\delta Z_{\delta_e}}{Z_{\delta_e}}}$$

Torpedo A : $$\frac{\delta R}{R} = \frac{\delta Z_{\delta_e} / Z_{\delta_e}}{28{\cdot}70 - \dfrac{\delta Z_{\delta_e}}{Z_{\delta_e}}}$$

Torpedo B : $$\frac{\delta R}{R} = \frac{\delta Z_{\delta_e} / Z_{\delta_e}}{-3{\cdot}38 - \dfrac{\delta Z_{\delta_e}}{Z_{\delta_e}}}.$$

Errors in M_{δ_e}

$$M_{\delta_e} \to M_{\delta_e} + \delta M_{\delta_e}$$

$$\frac{\delta R}{R} = \frac{\delta M_{\delta_e} / M_{\delta_e}}{\dfrac{Z_{\delta_e}}{Z_\alpha} \dfrac{M_\alpha}{M_{\delta_e}} - 1 - \dfrac{\delta M_{\delta_e}}{M_{\delta_e}}}$$

Torpedo A : $$\frac{\delta R}{R} = \frac{\delta M_{\delta_e} / M_{\delta_e}}{-0{\cdot}97 - \dfrac{\delta M_{\delta_e}}{M_{\delta_e}}}$$

Torpedo B : $$\frac{\delta R}{R} = \frac{\delta M_{\delta_e} / M_{\delta_e}}{1{\cdot}42 - \dfrac{\delta M_{\delta_e}}{M_{\delta_e}}}.$$

<u>Errors in Z_q</u> $\qquad Z_q \to Z_q + \delta Z_q$

$$\frac{\delta R}{R} = \frac{-M_\alpha Z_q}{M_q Z_\alpha - M_\alpha(m_1 V + Z_q)} \frac{\delta Z_q}{Z_q} = \frac{Z_q}{(m_1 V + Z_q)} \frac{G_0 - 1}{G_0} \frac{\delta Z_q}{Z_q},$$

where G_0 is the value of G, the margin of stability, when there are no errors in the stability derivatives.

Torpedo A : $\quad \dfrac{\delta R}{R} = -0 \cdot 04 \dfrac{\delta Z_q}{Z_q}$

Torpedo B : $\quad \dfrac{\delta R}{R} = +0 \cdot 91 \dfrac{\delta Z_q}{Z_q}$.

<u>Errors in M_q</u> $\qquad M_q \to M_q + \delta M_q$

$$\frac{\delta R}{R} = \frac{M_q Z_\alpha}{M_q Z_\alpha - M_\alpha(m_1 V + Z_q)} \frac{\delta M_q}{M_q} = \frac{1}{G_0} \frac{\delta M_q}{M_q}$$

Torpedo A : $\quad \dfrac{\delta R}{R} = +0 \cdot 99 \dfrac{\delta M_q}{M_q}$

Torpedo B : $\quad \dfrac{\delta R}{R} = +1 \cdot 80 \dfrac{\delta M_q}{M_q}$.

These results are plotted in the form percentage error in R against percentage error in C in Fig. 2 for Torpedo A, and in Fig. 3 for Torpedo B. It is clear from Fig. 2 that, for Torpedo A, errors only in M_q and M_{δ_e} are significant. It is therefore useful to study the variation of R when there are errors in M_q and M_{δ_e} simultaneously. The result for Torpedo A is

$$\frac{\delta R}{R} = \frac{+0 \cdot 99 \dfrac{\delta M_q}{M_q} - 1 \cdot 04 \dfrac{\delta M_{\delta_e}}{M_{\delta_e}}}{1 + 1 \cdot 04 \dfrac{\delta M_{\delta_e}}{M_{\delta_e}}}.$$

This can be plotted as a family of straight lines in the $\delta R/R - \delta M_q/M_q$ plane with $\delta M_{\delta_e}/M_{\delta_e}$ as parameter. From this it can be seen what ranges of errors (positive and negative) in M_{δ_e} and M_q are permissible for a given permissible range of error in R. This information is plotted in Fig. 4.

For Torpedo B errors in all stability derivatives are significant, and there is no point in considering simultaneous variations of two only.

3.2. *The Effect on the Margin of Stability.*—G was defined by equation (7) as

$$G_0 = 1 - \frac{M_\alpha(m_1 V + Z_q)}{Z_\alpha M_q},$$

where G_0 is the value of G when there are no errors in the stability derivatives. We are now interested in the value of G when errors in the stability derivatives exist, and not in the fractional change in G. The values of G_0 for the two torpedoes being considered are

Torpedo A : $\quad G_0 = +1 \cdot 011$

Torpedo B : $\quad G_0 = +0 \cdot 556$.

Errors in Z_α $$Z_\alpha \to Z_\alpha + \delta Z_\alpha$$

$$G = 1 + \frac{G_0 - 1}{1 + \dfrac{\delta Z_\alpha}{Z_\alpha}}$$

Torpedo A : $$G = 1 + \frac{0\cdot011}{1 + \dfrac{\delta Z_\alpha}{Z_\alpha}}$$

Torpedo B : $$G = 1 - \frac{0\cdot444}{1 + \dfrac{\delta Z_\alpha}{Z_\alpha}} .$$

Errors in M_α $$M_\alpha \to M_\alpha + \delta M_\alpha$$

$$G = G_0 + (G_0 - 1)\frac{\delta M_\alpha}{M_\alpha}$$

Torpedo A : $$G = 1\cdot011 + 0\cdot011\frac{\delta M_\alpha}{M_\alpha}$$

Torpedo B : $$G = 0\cdot556 - 0\cdot444\frac{\delta M_\alpha}{M_\alpha} .$$

Errors in Z_q $$Z_q \to Z_q + \delta Z_q$$

$$G = G_0 + \frac{Z_q}{m_1 V + Z_q}(G_0 - 1)\frac{\delta Z_q}{Z}$$

Torpedo A : $$G = 1\cdot011 - 0\cdot040\frac{\delta Z_q}{Z_q}$$

Torpedo B : $$G = 0\cdot556 + 0\cdot050\frac{\delta Z_q}{Z_q} .$$

Errors in M_q $$M_q \to M_q + \delta M_q$$

$$G = 1 + \frac{G_0 - 1}{1 + \dfrac{\delta M_q}{M_q}}$$

Torpedo A : $$G = 1 + \frac{0\cdot011}{1 + \dfrac{\delta M_q}{M_q}}$$

Torpedo B : $$G = 1 - \frac{0\cdot444}{1 + \dfrac{\delta M_q}{M_q}}\text{, which is the same variation as for } \frac{\delta Z_\alpha}{Z_\alpha} .$$

These results are plotted with $\delta C/C$ as a percentage in Fig. 5 for Torpedo A and in Fig. 6 for Torpedo B. It is obvious from the form of the equations that the variation of G with errors in the derivatives decreases as G_0 approaches unity and is in fact zero at $G_0 = 1$.

3.3. *The Effect on the Transient Motion of the Torpedo, Following a Disturbance.*—It was shown in Section 2 that the transient part of the solution for the angle of attack $\alpha(t)$ following a disturbance was the expression (5) :

$$\lambda_1 e^{\mu_1 t} + \lambda_2 e^{\mu_2 t} .$$

The transient solution for the depth z_0, or pitching rate $\dot{\theta}$ would be of the same form, with of course, different values of the constants λ_1 and λ_2. Real values of μ_1 and μ_2 will be associated with aperiodic motion, and imaginary values with oscillatory motion.

The effect of errors in the stability derivatives on the transient motion of the torpedo can be studied in two sub-sections :

(*a*) The effect of such errors on the decay constants μ_1 and μ_2

(*b*) The effect of such errors on the transient motion following one particular disturbance which will be taken as a step function input to the elevators.

3.3 (*a*).—*The effect of errors on the decay constants.*—The decay constants were defined by equations (4) and (6). It is obvious from these that there are two types of problem involved since errors in Z_q or M_z cause A_3 only to vary, while errors in Z_α or M_q cause both A_2 and A_3 to vary.

Errors in M_z $$M_z \to M_z + \delta M_z$$

Let μ be a root of the new equation (replacing equation (6))

$$A_1\mu^2 + A_2\mu + A_3 - (m_1 V + Z_q)M_z\frac{\delta M_z}{M_z} = 0 \, .$$

Put $\mu = y$ and $\delta M_z/M_z = x$, and this becomes the equation of a conic in the x-y plane. In conventional conic notation, it becomes

$$b_1 y^2 + 2g_1 x + 2f_1 y + c_1 = 0 \, ,$$

where
$$\begin{aligned} b_1 &= + A_1 = m_2 V J_y \\ 2g_1 &= - M_z(m_1 V + Z_q) \\ 2f_1 &= + A_2 = - m_2 V M_q - J_y Z_z \\ c_1 &= + A_3 = M_q Z_\alpha - M_z(m_1 V + Z_q) \, . \end{aligned}$$

The discriminant Δ is, in conic notation, $h_1^2 - a_1 b_1 = 0$. Hence the equation above represents a parabola, providing the conic is non-degenerate (the case where the conic is degenerate is discussed below). The parabola passes through the points $(0, \mu_1)$ and $(0, \mu_2)$ and its axis is parallel to the x axis. Its vertex has an x co-ordinate of

$$\frac{f_1^2 - b_1 c_1}{2b_1 g_1} = -1 - \frac{(m_2 V M_q - J_y Z_z)^2}{4m_2 V J_y M_z(m_1 V + Z_q)} \, .$$

The value of the decay constants for any given error in M_z, say δM_z^*, are the values of y at which the line $x = \delta M_z^*/M_z$ meets the parabola.

The parabola cuts the x axis at the point $x = G_0/(1 - G_0)$, $y = 0$, where G_0 is the margin of stability calculated when no errors exist in any derivative. With this value of x, the torpedo is marginally dynamically stable. Moreover, the nearer G_0 is to unity the smaller is the change in the decay constants for any given error. At $G_0 = 1$, the coefficient g_1 in the equation of the parabola disappears, and this is the condition for the parabola to degenerate into a parallel line-pair in the direction of the x axis, which implies no change at all in the decay constants for errors in M_z. We assume that when no errors exist, the torpedo is dynamically stable, that is $G_0 > 0$ and μ_1 and μ_2 negative. It follows that the parabola faces right or left according as $G_0 \gtrless 1$.

The parabola is plotted in Fig. 7 for Torpedo A, and in Fig. 8 for Torpedo B. It should be noticed that the horizontal scales of these diagrams are in units of $\delta M_z/M_z$ and not $(\delta M_z/M_z)$ per cent as in previous diagrams. The variations of the decay constants are greater for

Torpedo B than for Torpedo A, as is to be expected, since G_0 is nearer unity for Torpedo A. In fact, for Torpedo A, over the range $\left|\frac{\delta M_\alpha}{M_\alpha}\right| \leqslant 0{\cdot}2$ (*i.e.*, $\pm$ 20 per cent error), there is no noticeable change in the decay constants. For Torpedo B the change in the decay constants for the same range of $\delta M_\alpha / M_\alpha$ is noticeable but not significant.

The torpedo is dynamically stable or unstable according as μ_1 and μ_2 have negative or positive real parts. When μ_1 and μ_2 become imaginary (*i.e.*, in the region of the diagram past the vertex of the parabola), the motion hitherto aperiodic becomes oscillatory. That the oscillatory motion is, in fact, stable can be easily checked.

Errors in Z_q $\qquad\qquad Z_q \to Z_q + \delta Z_q$

Let μ be a root of the new equation

$$A_1\mu^2 + A_2\mu + A_3 - M_\alpha Z_q \frac{\delta Z_q}{Z_q} = 0 .$$

Put $\mu = y$ and $\delta Z_q / Z_q = x$ and we can write this in conic notation as before

$$b_2 y^2 + 2g_2 x + 2f_2 y + c_2 = 0 ,$$

where

$$\begin{aligned} b_2 &= +A_1 = m_2 V J_y \\ 2g_2 &= -M_\alpha Z_q \\ 2f_2 &= +A_2 = -m_2 V M_q - J_y Z_\alpha \\ c_2 &= +A_3 = M_q Z_\alpha - M_\alpha (m_1 V + Z_q) . \end{aligned}$$

This is, again, a parabola passing through $(0, \mu_1)$ and $(0, \mu_2)$. The x co-ordinate of the vertex is now

$$\frac{f_2^2 - b_2 c_2}{2 b_2 g_2} = \left[-1 - \frac{(m_2 V M_q - J_y Z_\alpha)^2}{4 m_2 V J_y M_\alpha (m_1 V + Z_q)} \right] \frac{m_1 V + Z_q}{Z_q} .$$

It will meet the x axis where

$$x = \frac{G_0}{1 - G_0} \, \frac{m_1 V + Z_q}{Z_q} .$$

It is in fact the same parabola as before, but with the horizontal scale multiplied by a factor $(m_1 V + Z_q)/Z_q$. Minimum variation again occurs when $G_0 = 1$, when the parabola degenerates as before. The parabola is plotted in Fig. 7 for Torpedo A and Fig. 8 for Torpedo B. In both cases the variation of the decay constants is a little greater than for the M_α case but it is still negligible for Torpedo A and not very significant for Torpedo B in the range $\left|\frac{\delta Z_q}{Z_q}\right| \leqslant 0{\cdot}2$.

Errors in M_q $\qquad\qquad M_q \to M_q + \delta M_q$

Let μ be a root of the new equation

$$A_1\mu^2 + \left(A_2 - m_2 V M_q \frac{\delta M_q}{M_q}\right)\mu + A_3 + Z_\alpha M_q \frac{\delta M_q}{M_q} = 0 .$$

Put $y = \mu$ and $x = \delta M_q / M_q$. In conic notation the equation becomes

$$2h_3 xy + b_3 y^2 + 2g_3 x + 2f_3 y + c_3 = 0 , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where

$$\left.\begin{aligned} 2h_3 &= -m_2 V M_q \\ b_3 &= +A_1 = m_2 V J_y \\ 2g_3 &= +Z_\alpha M_q \\ 2f_3 &= +A_2 = -J_y Z_\alpha - m_2 V M_q \\ c_3 &= +A_3 = M_q Z_\alpha - M_\alpha (m_1 V + Z_q) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

The discriminant $\Delta = h_3^2 - a_3b_3 = h_3^2 > 0$, so the equation represents a conic which, if non-degenerate, is a hyperbola. (The case when the conic is degenerate will be discussed below). The equation of the asymptotes is got from this equation by adding a constant $\varkappa$ such that

$$\begin{vmatrix} \cdot & h_3 & g_3 \\ h_3 & b_3 & f_3 \\ g_3 & f_3 & c_3 + \varkappa \end{vmatrix} = 0 .$$

Solving for $\varkappa$ we get

$$c_3 + \varkappa = 2g_3\left(\frac{4f_3h_3 - 2b_3g_3}{4h_3^2}\right) = 2g_3 ,$$

since $\qquad 4f_3h_3 - 2b_3g_3 = 4h_3^2$ from (13).

The asymptote pair has, therefore, the equation

$$2h_3xy + b_3y^2 + 2g_3x + 2f_3y + 2g_3 = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

The absence of a term in x^2 shows that one of the asymptotes is parallel to the x axis. The slope of the other one is therefore the tangent of the angle between them and is

$$\pm\frac{2\sqrt{(h_3^2 - a_3b_3)}}{a_3 + b_3} = \pm\frac{M_q}{J_y} .$$

From (14) we see that the point $(-1, 0)$ lies on the asymptote pair, and since the horizontal asymptote is certainly not $y = 0$, the point $(-1, 0)$ necessarily lies on the sloping asymptote, whose equation is therefore

$$y = \pm\frac{M_q}{J_y}(1 + x) .$$

Since the hyperbola passes through the points $(0, \mu_1)$ and $(0, \mu_2)$ where μ_1 and μ_2 are negative, this asymptote must have a negative gradient, whence its equation is

$$y = \frac{M_q}{J_y}(1 + x) ,$$

M_q being negative for all conventional torpedoes. The equation of the other asymptote is found by differentiating equation (14) and finding the value of y for which dy/dx vanishes. It is

$$y = -\frac{g_3}{h_3} = \frac{Z_w}{m_1V} .$$

The horizontal asymptote has therefore the equation

$$y = \frac{Z_w}{m_1V} .$$

We note that the asymptotes intersect at (x^*, y^*), where

$$x^* = \frac{J_yZ_w - m_1VM_q}{m_1VM_q} .$$

We can now draw the asymptotes directly, and we know, moreover, two points on the hyperbola, namely, $(0, \mu_1)$ and $(0, \mu_2)$. There is one other point of interest on the hyperbola. From equation (12) the x axis cuts the hyperbola where

$$x = \frac{-c_3}{2g_3} = -G_0 .$$

There are four possible configurations of the hyperbola depending on whether $x^* \gtrless 0$ and $G_0 \gtrless 1$. These are shown in Fig. 9. If we use the fact that the intercepts on any straight line cut off between a hyperbola and its asymptotes are equal, it is possible to sketch in the hyperbola with reasonable accuracy from a knowledge of its asymptotes, the points $(0, \mu_1)$, $(0, \mu_2)$ and $(-G_m, 0)$, which are known to lie on it. In the case $G_m < 1$, as $G_m \to 1$, the rate of variation of one decay constant decreases, while that of the other increases to the slope of the sloping asymptote. In the case $G_m > 1$ it is clear that the variation of both decay constants decreases as G_m approaches unity. If $G_m = 1$, the hyperbola degenerates into its asymptotes, and only one decay constant varies.

The hyperbola for Torpedo A is shown in Fig. 10, and for Torpedo B in Fig. 11, and the stability regions are shown for each. It is easily proved that the region of oscillatory motion is a region of stable motion. It is interesting to note that when $G_m > 1$ it is impossible to reach a condition of oscillatory motion of the body by altering M_q only.

It is clear from these Figures that errors in M_q are far more significant as regards the decay constants, than are errors in Z_w and M_w. In fact an error of $-$ 60 per cent in M_q would cause Torpedo A to oscillate, and Torpedo B to become dynamically unstable.

<u>Errors in Z_w.</u> $\qquad Z_w' = Z_w + \delta Z_w$

Let μ be a root of the new equation

$$A_0\mu^2 + \left(A_1 - J Z_w \frac{\delta Z_w}{Z_w}\right)\mu + A_2 - M_q Z_w \frac{\delta Z_w}{Z_w} = 0 .$$

Putting $\mu = y$ and $\delta Z_w / Z_w = x$, this equation becomes, in conic notation,

$$2h_1 xy + b_1 y^2 + 2g_1 x + 2f_1 y + c_1 = 0 ,$$

where

$$2h_1 = -J Z_w$$

$$b_1 = A_0 = m_1 V J$$

$$2g_1 = -M_q Z_w$$

$$2f_1 = A_1 = -J Z_w - m_1 V M_q$$

$$c_1 = A_2 = M_q Z_w - M_w(m_1 V + Z_q) .$$

This is again a hyperbola and, in the same way as before, we find that the asymptotes have the equations

$$y = -\frac{M_q}{J} \qquad \text{(horizontal asymptote)}$$

$$y = \frac{Z_w}{m_1 V}(1 + x) \qquad \text{(sloping asymptote)} .$$

They intersect in the point (x^*, y^*), where

$$x^* = \frac{m_1 V M_q - J Z_w}{J Z_w} .$$

Since the x axis cuts the hyperbola at $x = -c_1/2g_1 = -G_m$ as before, the remarks made about the significance of having a value of G_m close to unity still apply. The four configurations shown in Fig. 9 also apply, if the new expression for x^* is used. These hyperbolae are plotted in Fig. 10 for Torpedo A and in Fig. 11 for Torpedo B. The variations in the decay constants are still large but not so critically as they were for errors in M_q, particularly as regards measuring accuracy, since accuracy in measuring Z_w is far easier to achieve than accuracy in measuring M_q. For Torpedo A an error of $-$ 100 per cent would be necessary to cause instability and an error of + 240 per cent to cause oscillatory motion. For Torpedo B, instability would occur when Z_w

had an error of -60 per cent. For both torpedoes, x^* is positive for the Z_α case and negative for the M_q case. This implies that the sloping asymptote has a less steep gradient in the Z_α case than in the M_q case, and that the variations in the decay constants are correspondingly less.

3.3. (*b*). *The effect of errors on the transient motion for one particular disturbance.*—The disturbance will be taken as a step function input on the elevators. The subsequent solution for the angle of attack will be studied. The relevant equation is equation (3), where $\delta_e(t)$ is now a step function of magnitude δ_e^*. Then,

$$\bar{\delta}_e(p) = \frac{1}{p}\,\delta_e^* \,,$$

and equation (3) gives,

$$\frac{\bar{\alpha}(p)}{\delta_e^*} = \frac{J_y Z_{\delta_e} p + M_{\delta_e}(m_1 V + Z_q) - M_q Z_{\delta_e}}{m_2 V J_y p(p - \mu_1)(p - \mu_2)}$$

by the definition of μ_1 and μ_2. Splitting the right-hand side into partial fractions we have

$$\frac{\bar{\alpha}(p)}{\delta_e^*} = \frac{\lambda_3}{p} + \frac{\lambda_1}{p - \mu_1} + \frac{\lambda_2}{p - \mu_2}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (15)$$

where

$$\left.\begin{aligned} \lambda_3 &= \frac{M_{\delta_e}(m_1 V + Z_q) - M_q Z_{\delta_e}}{m_2 V J_y \mu_1 \mu_2} \\ \lambda_1 &= \frac{J_y Z_{\delta_e}\mu_1 + M_{\delta_e}(m_1 V + Z_q) - M_q Z_{\delta_e}}{m_2 V J_y \mu_1(\mu_1 - \mu_2)} \\ \lambda_2 &= \frac{J_y Z_{\delta_e}\mu_2 + M_{\delta_e}(m_1 V + Z_q) - M_q Z_{\delta_e}}{m_2 V J_y \mu_2(\mu_2 - \mu_1)} \end{aligned}\right\} \quad \ldots \quad \ldots \qquad (16)$$

Inverse Laplace-transforming equation (15) gives

$$\frac{\alpha(t)}{\delta_e^*} = \lambda_3 + \lambda_1 e^{\mu_1 t} + \lambda_2 e^{\mu_2 t} \,.$$

Since we are interested only in the transient solution, and not in the steady-state solution (which is λ_3), we divide by λ_3 to get finally,

$$\left.\begin{aligned} \frac{\alpha(t)}{\lambda_3 \delta_e^*} &= 1 + \lambda_1' e^{\mu_1 t} + \lambda_2' e^{\mu_2 t} \\ \text{where} \quad \lambda_1' &= \frac{\lambda_1}{\lambda_3} = \left[\frac{J_y Z_{\delta_e}\mu_1}{M_{\delta_e}(m_1 V + Z_q) - M_q Z_{\delta_e}} + 1\right]\frac{\mu_2}{\mu_1 - \mu_2} \\ \lambda_2' &= \frac{\lambda_2}{\lambda_3} = \left[\frac{J_y Z_{\delta_e}\mu_2}{M_{\delta_e}(m_1 V + Z_q) - M_q Z_{\delta_e}} + 1\right]\frac{\mu_1}{\mu_2 - \mu_1} \end{aligned}\right\} \quad \ldots \quad \ldots \qquad (17)$$

μ_1 and μ_2 are affected by errors in Z_α, M_α, Z_q and M_q as already shown. λ_1' and λ_2' are affected by errors in all six derivatives. It is therefore possible to study how the solution (16) varies with errors in each of the six stability derivatives, one at a time. This has been done for three values of error in each derivative, namely 0, $\pm$ 50 per cent for the rotary derivatives Z_q and M_q, and 0, $\pm$ 20 per cent for the others. The results for Torpedo A are contained in Fig. 12, and for Torpedo B in Fig. 13. The time for the ordinate to reach 95 per cent of its final value is marked in each case. Errors in Z_{δ_e} and M_{δ_e} do not affect either torpedo noticeably. For the remaining derivatives, errors appear to affect Torpedo B more adversely than they do Torpedo A particularly in the case of the rotary derivatives Z_q and M_q. An error of $-$ 50 per cent in M_q causes a substantial change in the motion of Torpedo B. It should be noticed that the time to reach 95 per cent of the final value is less for Torpedo A than for Torpedo B ; this is to be expected since Torpedo A has a larger margin of stability.

4. *Summary and Conclusions.*—In this report, the extent to which the dynamic behaviour of the torpedo is sensitive to changes in its stability derivatives has been investigated. Attention has necessarily been confined to certain well defined aspects of dynamic behaviour. These aspects were the radius of turn for a given elevator angle, the margin of stability, the decay constants of disturbed motion, and the motion following a particular disturbance, namely, a step function input to the elevators. It is not too unreasonable to suppose that these aspects are broadly representative of dynamic behaviour. It must be admitted, however, that the theoretical results apply to an uncontrolled torpedo. Nevertheless, it should be noted that according to the Table, the margin of stability indicates the ease with which a control system for a homing torpedo can be designed.

The results obtained in particular cases, namely, Torpedo A and Torpedo B which have been used as illustrative examples, may be summarised as follows : The radius of turn per elevator angle of Torpedo A is very susceptible to errors in M_{δ_e} and M_q ; that of Torpedo B is very susceptible to errors in all derivatives except perhaps Z_{δ_e}. The margin of stability G, for Torpedo A varies very little with errors in the stability derivatives. For Torpedo B, G varies rapidly with errors in Z_α, M_q and M_α. For both torpedoes, the decay constants vary much more with errors in Z_α and M_q than with errors in M_α and Z_q. This tendency is reflected in the effect of errors on the solution for angle of attack following a step function input to the elevators, but it is not as pronounced as one would expect, presumably due to the effects of the errors on the coefficients λ_1' and λ_2'. For Torpedo A, the variation of the solution is small for all feasible errors. This is not so for Torpedo B, the variations due to errors in Z_α and M_q being rather severe.

In view of the complexity of the concept of dynamic behaviour and the number of parameters involved, it is difficult to draw general conclusions. It does seem clear, however, that the susceptibility of torpedo performance to changes or errors in the stability derivatives depends to a great extent on the margin of stability. The effect of errors is, in most respects, at a minimum when $G_0 = 1$, that is, when the torpedo is marginally statically stable.

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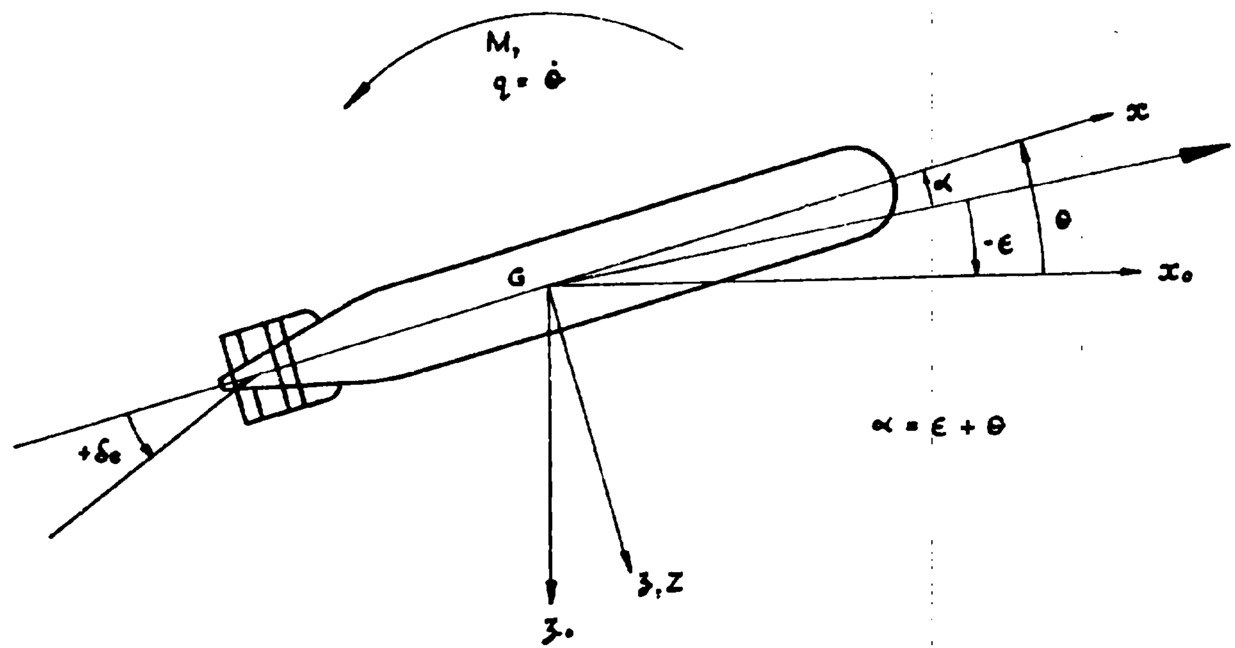


FIG. 1. Sign convention in pitch plane.

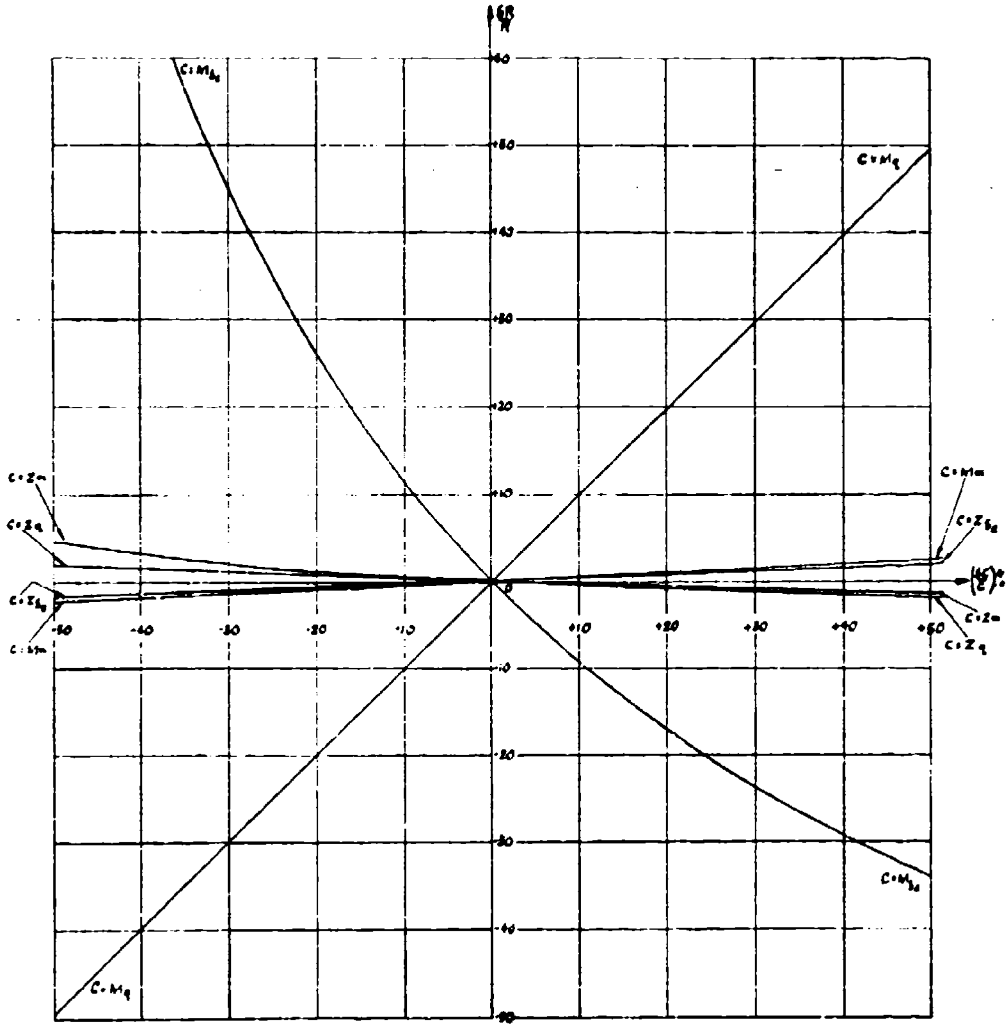


FIG. 2. Percentage error in radius of turn R against percentage error in hydrodynamic coefficients C (Torpedo A).

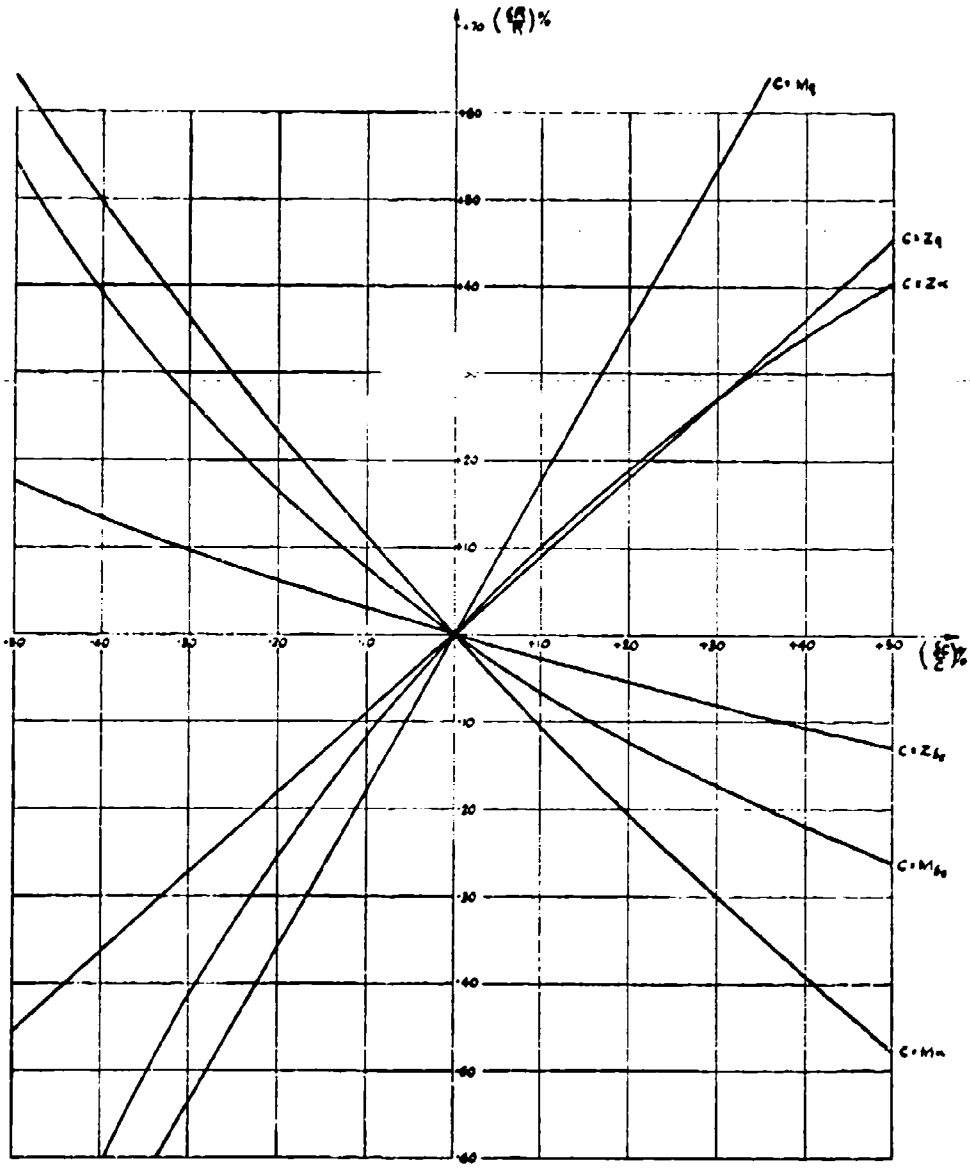


FIG. 3. Percentage error in radius of turn R against percentage error in hydrodynamic coefficients C (Torpedo A).

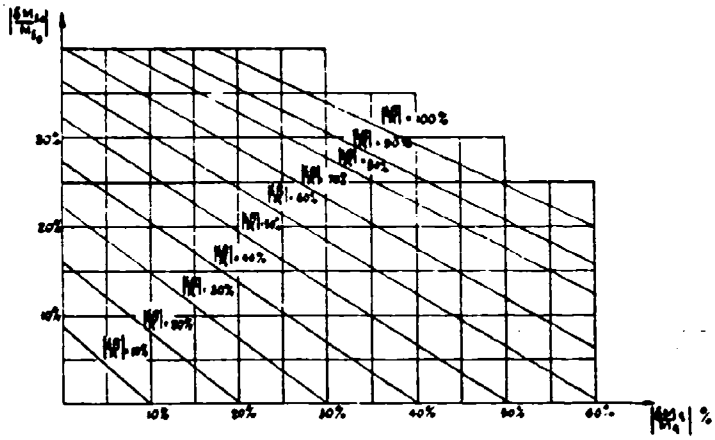


FIG. 4. Ranges for errors in M_{δ_e} and M_q for various permissible errors in R (Torpedo A).

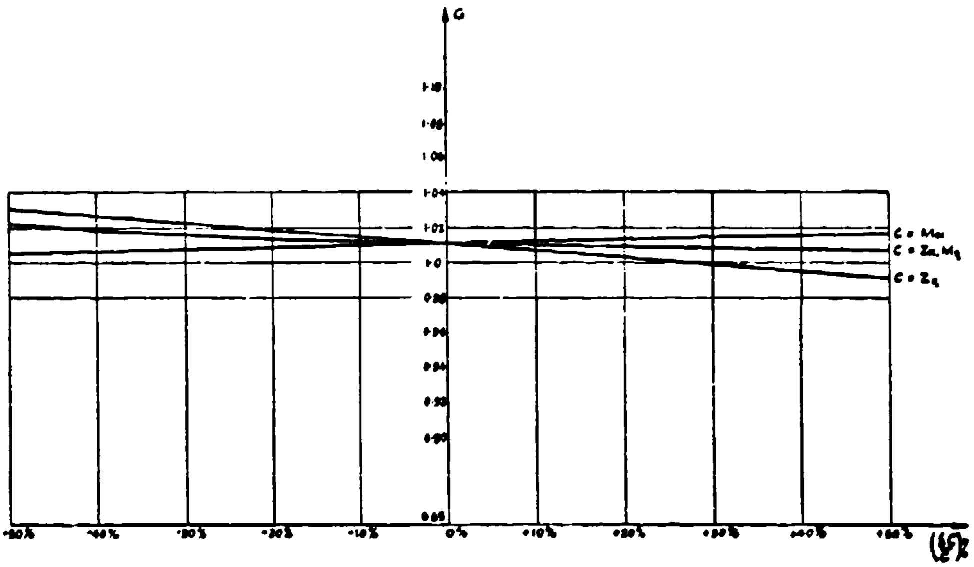


FIG. 5. The variation of G, the margin of stability with errors in the stability coefficients (Torpedo A).

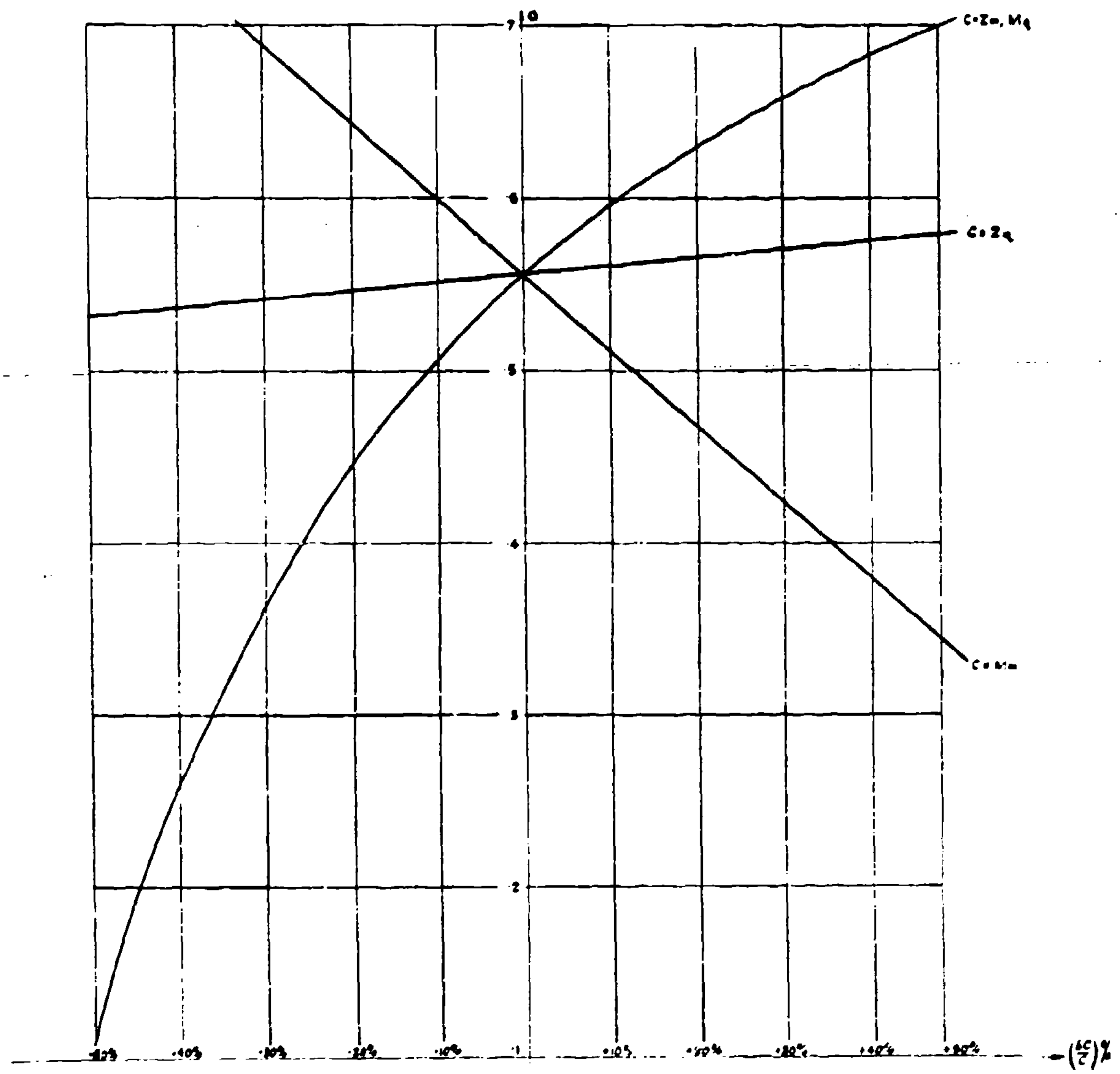


FIG. 6. The variation of G, the margin of stability, with errors in the stability derivatives (Torpedo B).

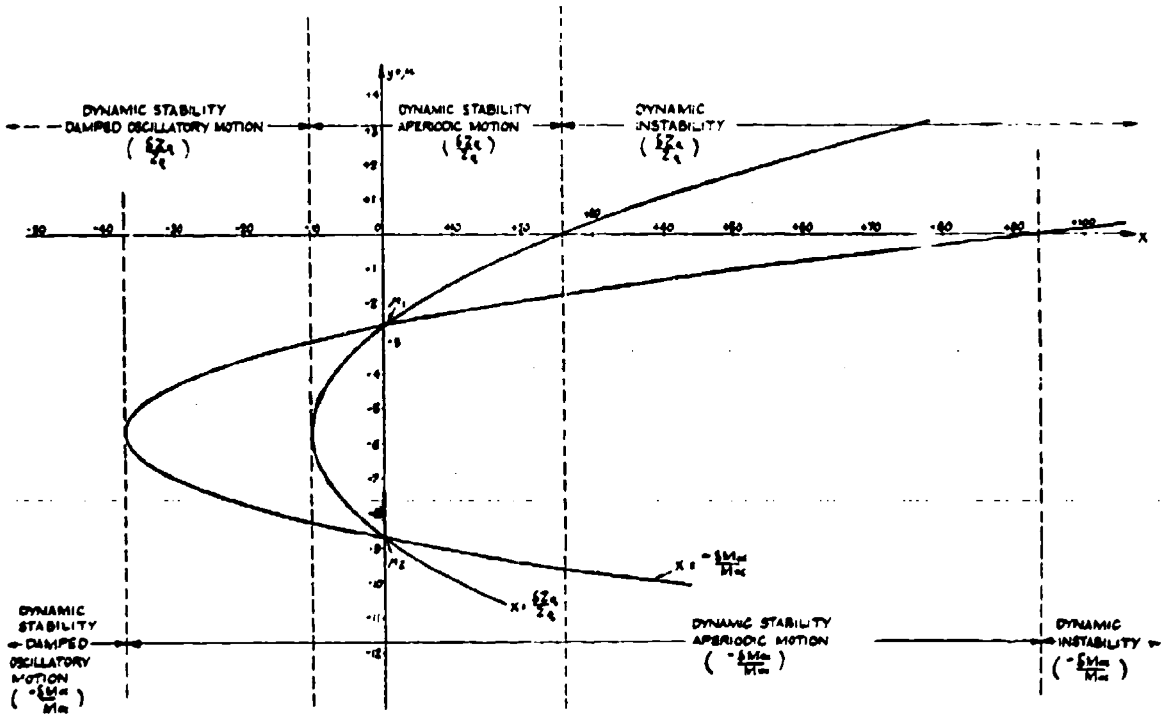


FIG. 7. The variation of the decay constants μ_1, μ_2, with $\delta Z_q/Z_q$, $\delta M_\alpha/M_\alpha$ (Torpedo A).

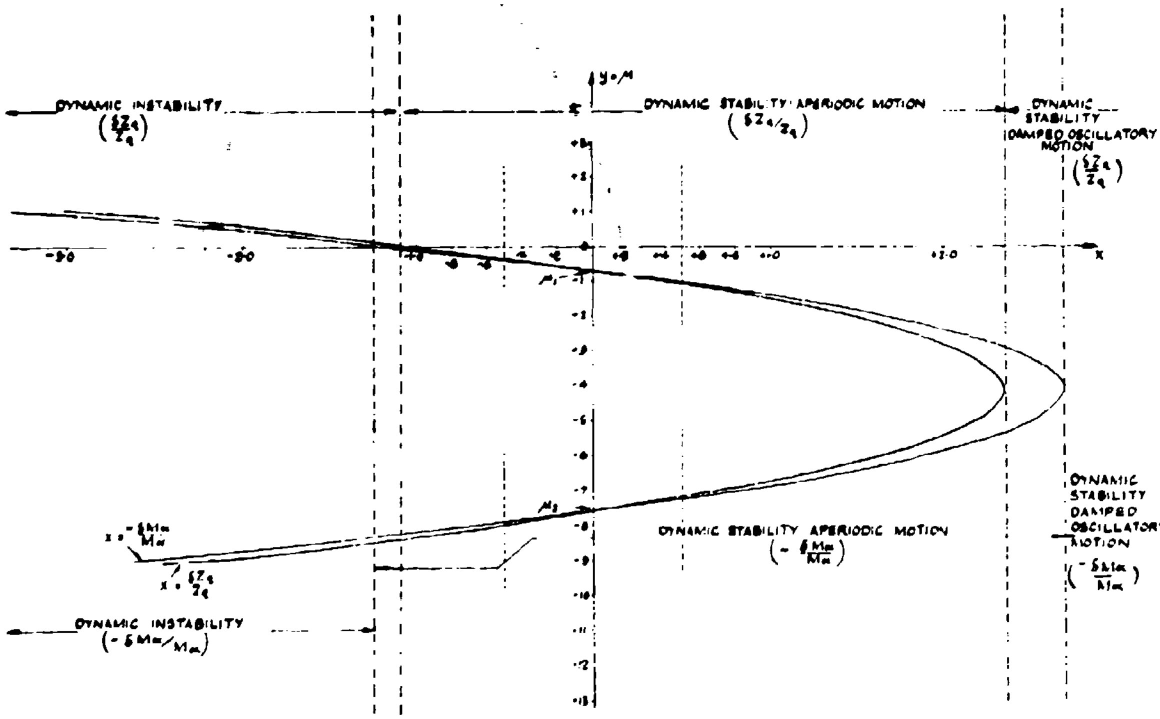


FIG. 8. The variation of the decay constants μ_1, μ_2, with errors in Z_q and M_α (Torpedo B).

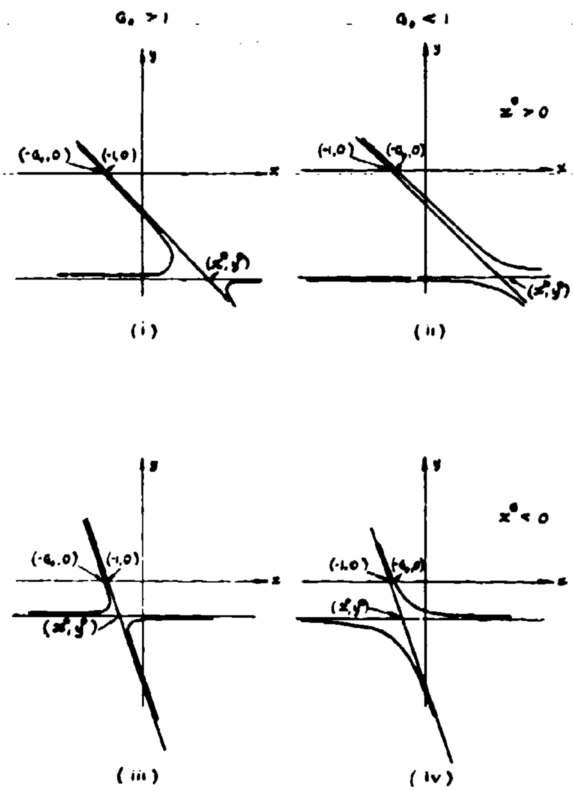


FIG. 9.

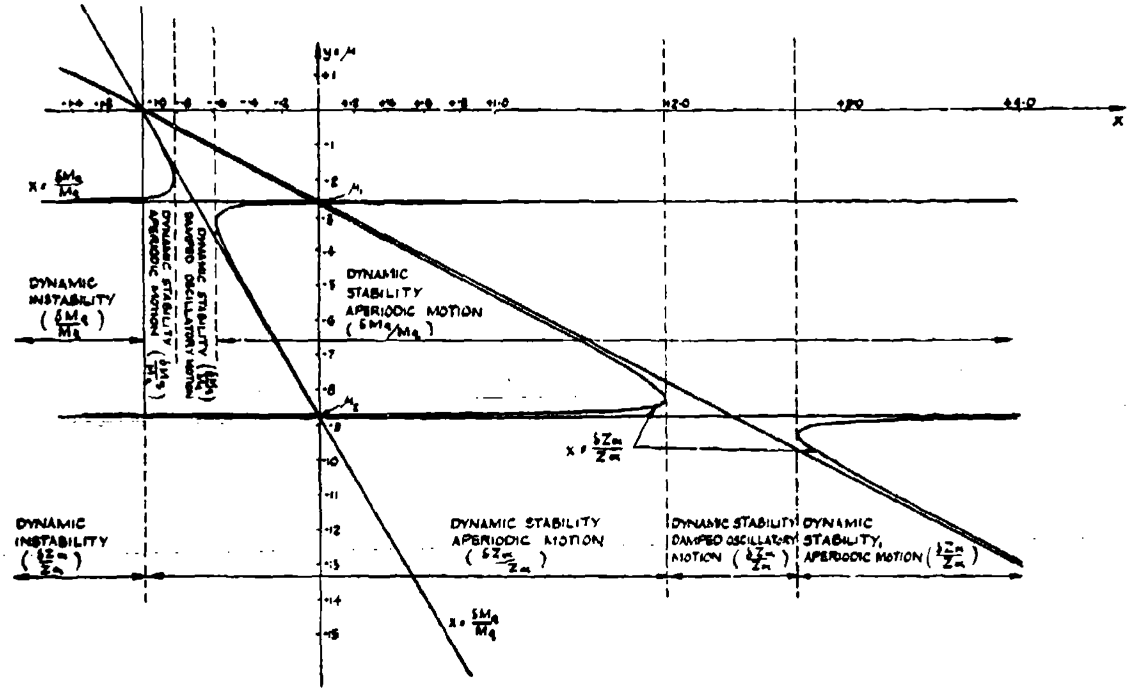


FIG. 10. Variation of decay constants μ_1 and μ_2 with $\delta Z_\alpha / Z_\alpha$ and $\delta M_q / M_q$ (Torpedo A).

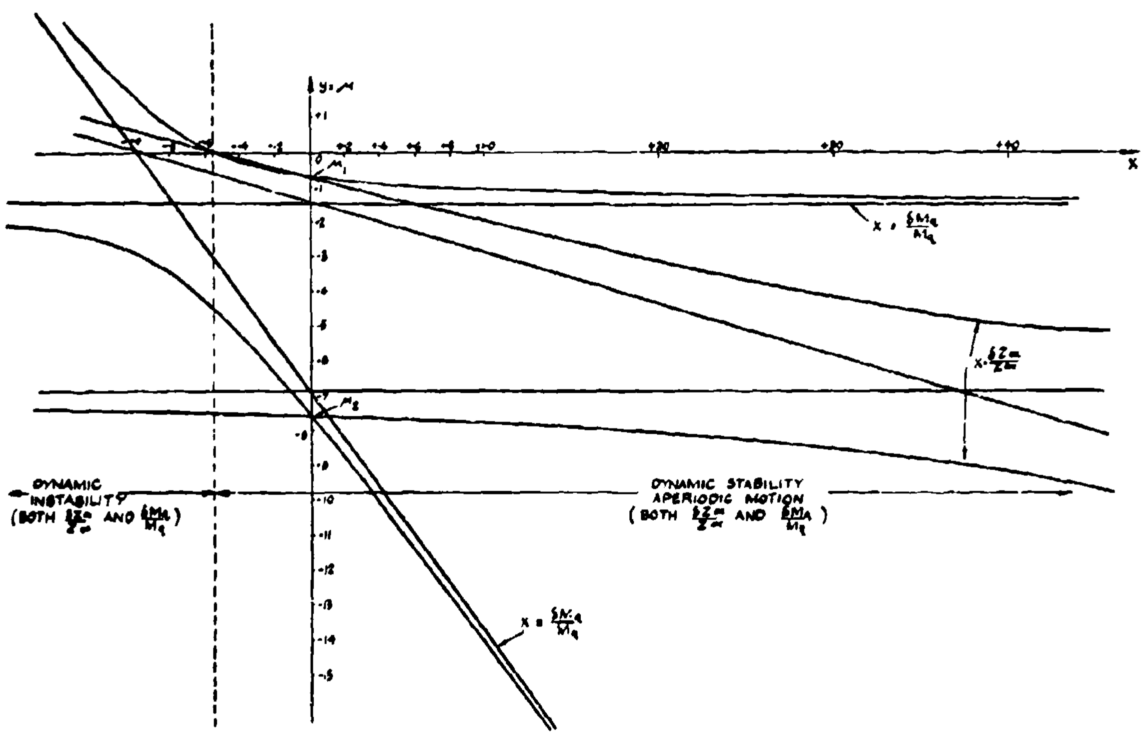
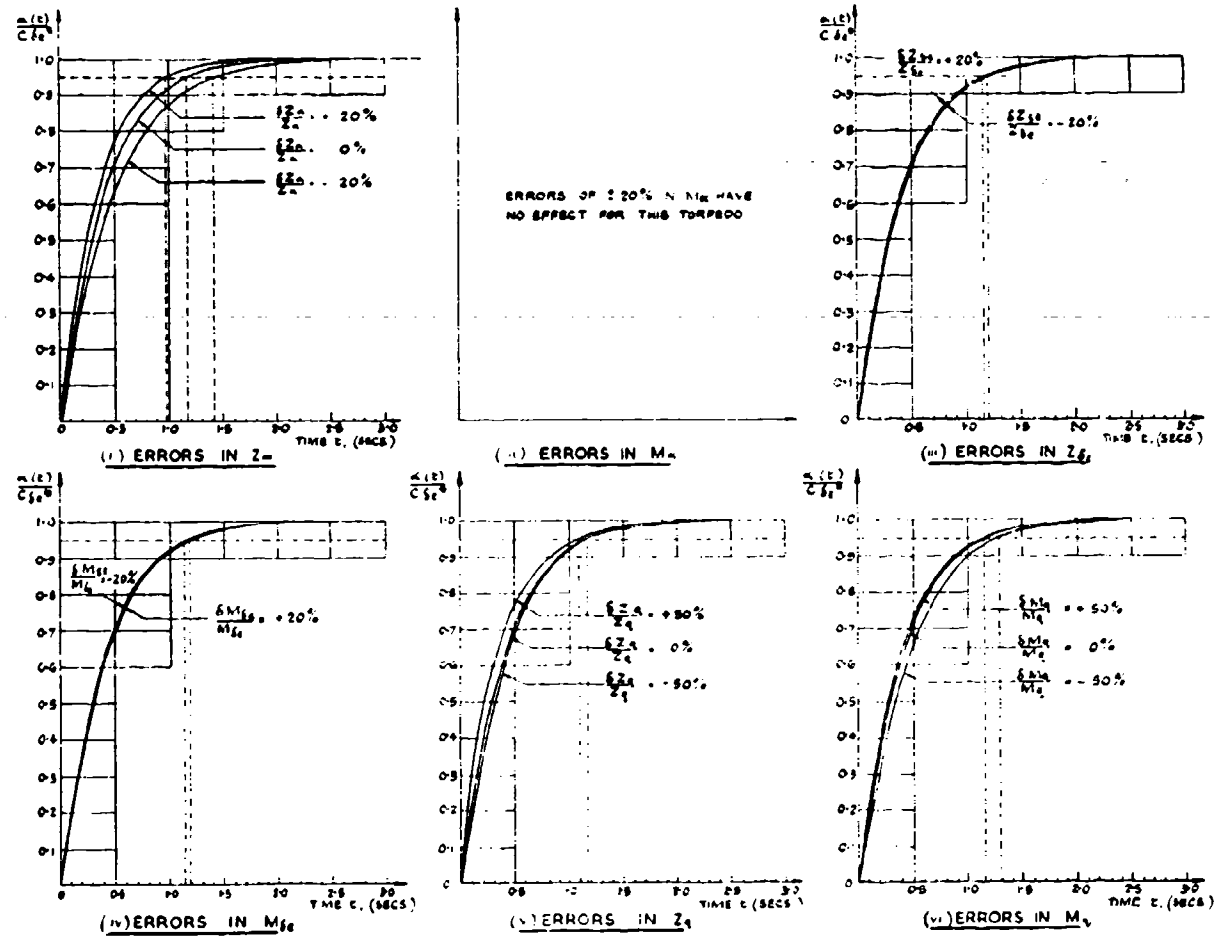


FIG. 11. Variation of decay constants μ_1 and μ_2 with $\delta Z_\alpha / Z_\alpha$ and $\delta M_q / M_q$ (Torpedo B).

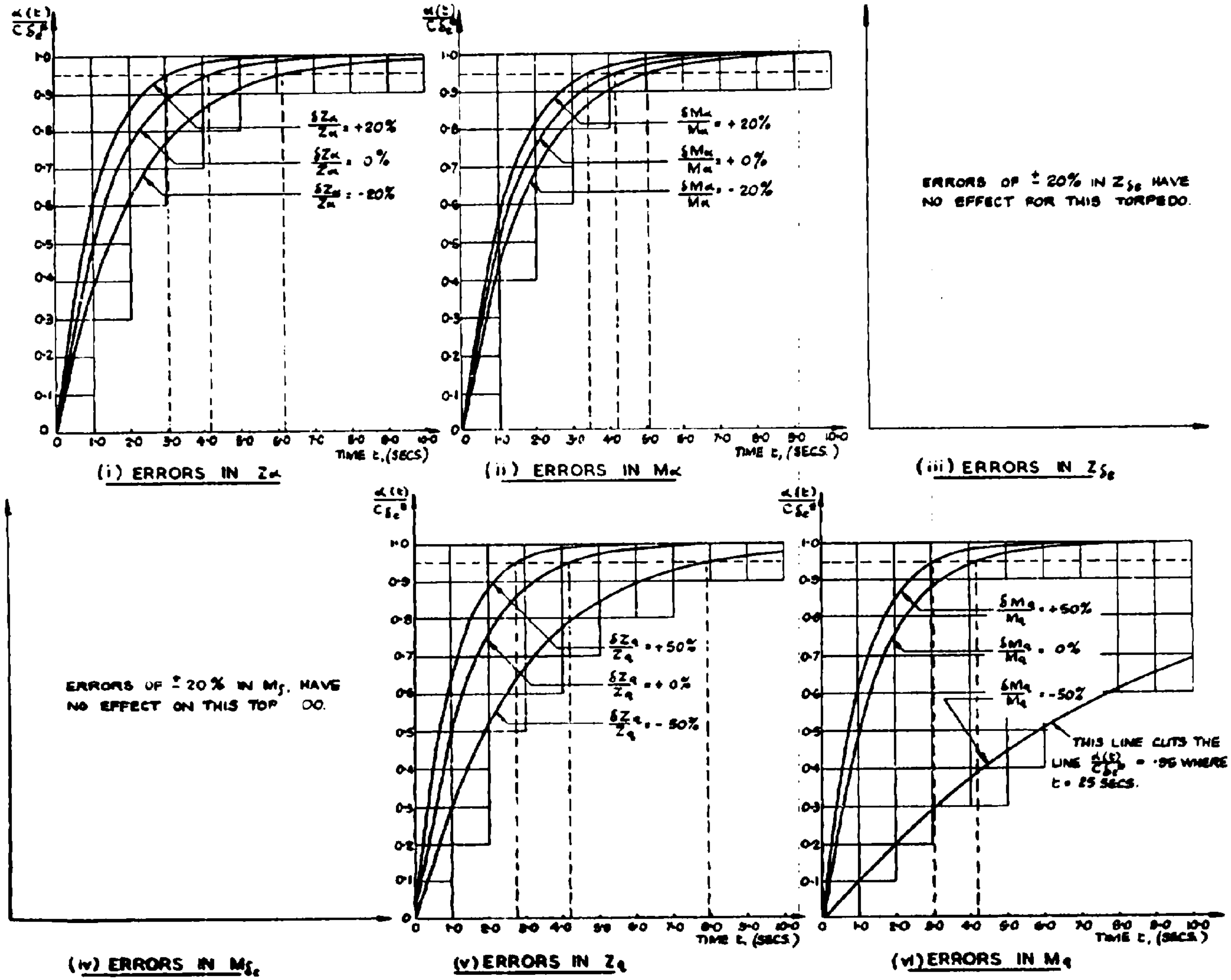


$\alpha(t)$ = ANGLE OF ATTACK FOLLOWING STEP FUNCTION INPUT ON ELEVATORS

δ_e^* = MAGNITUDE OF THE STEP FUNCTION

C = STEADY STATE VALUE OF ANGLE OF ATTACK

FIG. 12. The effect of errors on torpedo motion for Torpedo A.



$\alpha(t)$ = ANGLE OF ATTACK FOLLOWING STEP FUNCTION INPUT ON ELEVATORS.
δ_e^0 = MAGNITUDE OF THE STEP FUNCTION.
C = STEADY STATE VALUE OF ANGLE OF ATTACK.

FIG. 13. The effect of errors on torpedo motion for Torpedo B.

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